CERN/PS 96-36 (DI)
27.9.1996

TRANSIT TIME FOR THIRD ORDER RESONANCE EXTRACTION

M. Pullia *

Abstract

An important spin-off from accelerators is the use of synchrotrons for cancer therapy. A precise control of the extraction from the synchrotron is needed to satisfy the medical specifications and this has led to a renewed interest in the basic theory of third-order resonance extraction. In the present paper, a complete description of the transit time in the resonance (the time between a particle becoming unstable and reaching the electrostatic septum) is developed as a basis for future work predicting spill shapes and the influence of power supply ripple. The transit time is evaluated for constant tune and for a slowly varying tune. Both cases are subdivided into particles that start close to the stable point on the extraction separatrix and particles that start far from it somewhere along the side of the outermost stable triangle. Of the four expressions derived, the last turns out to be a good approximation for all cases. The analytic expressions are checked against numerical simulation and are shown to be correct to within a few percent.

* M. Pullia is a member of the TERA Foundation and an unpaid Associate of PPE Division, CERN, currently working on the Proton-Ion Medical Machine Study hosted by the PS Division,

1. INTRODUCTION

The evaluation of proton and light ion beams for cancer therapy is now a main issue in medical research and, as a consequence, an important spin-off for accelerator technology. The key issues for synchrotron designers are the precise control of the 3rd order resonance extraction to meet the medical specifications for spill times, intensities and uniformity and the design of a synchrotron that works reliably in a hospital environment, as opposed to a research laboratory. In the present report, analytic expressions are derived for the transit time in the resonance, that is the time between a particle becoming unstable and reaching the electrostatic extraction septum. This is regarded as an essential foundation for future work on the prediction of spill shapes, the influence of power supply ripple and possibly improvements in the efficiency of feedback systems for stabilizing the spill.

The motion of the particle is described by the 'Kobayashi' hamiltonian, H_K, see Ref. 1 by Barton. This is an approximate hamiltonian, chosen because it can be manipulated relatively easily and because numerical simulations show that the final expressions describe the physics to an accuracy of a few percent.

$$H_K = \frac{\varepsilon}{2}(X_K^2 + X'^2_K) + \frac{S}{4}(3X_K X'^2_K - X_K^3) \qquad (1)$$

where X_K and X'_K are the particle coordinates in normalized phase space [$m^{1/2}$], $\varepsilon = 6\pi\delta Q = 6\pi(Q_{part} - Q_{res})$, where Q_{part} is the tune of the particle and Q_{res} is the tune of the resonance, and S is the normalized sextupole strength [$m^{-1/2}$]. A brief reminder of the meaning of equation (1) is given in Appendix A and the geometry of the phase space is shown in Figure 1.

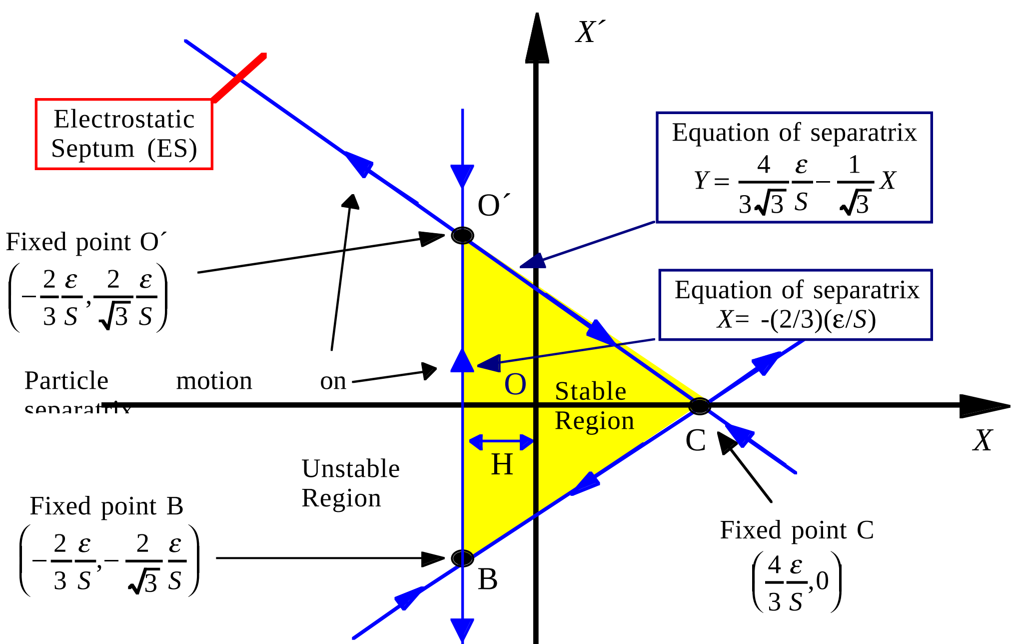

Figure 1. Phase space as described by the 'Kobayashi' hamiltonian.

When making an extraction, the stable area in Figure 1 is shrunk over a period of the order of 0.5 s, whereas the revolution time in the machine is typically of the order of 0.5 μs and the transit time of the order of 250 μs. Thus, particles that cross the limiting the following, the particles along the separatrix B to O´ in Figure 1 will be considered. The proximity of the extraction trajectories to the separatrices B to O´ and O´ to ES means that the influence of the stable point O´ will be strong and the particles will spend most of their time in the vicinity of this point. It is convenient therefore to change the coordinate system and translate the origin to the upper corner O´ of the stable triangle $\left(-\frac{2}{3}\frac{\varepsilon}{S}, \frac{2}{\sqrt{3}}\frac{\varepsilon}{S}\right)$, as represented in Figures 1 and 2. The hamiltonian $\overline{H}_K$ [m] in the new coordinates is given by:

$$\overline{H}_K = \frac{S}{4}(4H^3 + 6H\overline{X}^2 + 6\sqrt{3}H\overline{X}\,\overline{X}' + 3\overline{X}\,\overline{X}'^2 - \overline{X}^3) \qquad (2)$$

where $H = \frac{2}{3}\frac{\varepsilon}{S}$ [m$^{1/2}$] is the apothem of the triangle. To simplify the notation, from now on we will omit the bar over X and X' and we will rename X' with Y. Thus the new hamiltonian becomes:

$$\overline{H}_K = \frac{S}{4}(4H^3 + 6HX^2 + 6\sqrt{3}HX\,Y + 3X\,Y^2 - X^3) \qquad (3)$$

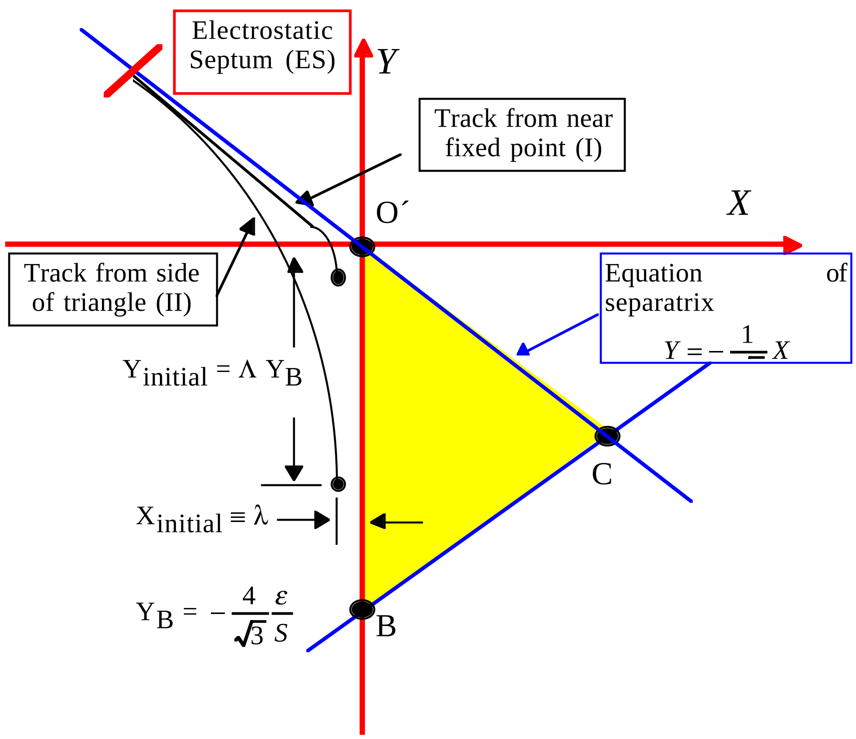

Figure 2. Change of origin and extraction trajectories.

The Kobayashi hamiltonian has been derived by approximating the phase-space displacement of a particle every third revolution of the accelerator, and considering the time of 3 turns as the infinitesimal increment for time. Thus, in the following, the unit of measure of the time is 3 revolution periods of the machine, $dt \equiv 3T_{REV}$ but is dimensionless (see Appendix A). The times derived from the following formulas, multiplied by 3, will thus give the number of turns necessary to reach the electrostatic septum.

The equations of motion in the new reference frame are given by:

$$\frac{dX}{dt} = \frac{\partial \overline{H}_K}{\partial Y} = \frac{S}{4}\left(6\sqrt{3}HX + 6XY\right) \qquad (a)$$

$$\frac{dY}{dt} = -\frac{\partial \overline{H}_K}{\partial X} = -\frac{S}{4}\left(12HX + 6\sqrt{3}HY + 3Y^2 - 3X^2\right) \qquad (b) \tag{4}$$

The extraction time for a particle will be worked out integrating the equations of motion between the starting position and the ES.

2. EXTRACTION TIME WITH CONSTANT PARAMETERS

If δQ, S and $\Delta p/p$ are constant during the extraction time, the particle will follow a trajectory with constant $\overline{H}_K$. Thus it is possible to work out the trajectory equation by equating the general hamiltonian to the value that it assumes with the initial conditions. From (3):

$$\frac{S}{4}(4H^3 + 6HX^2 + 6\sqrt{3}HXY + 3XY^2 - X^3) = $$
$$= \frac{S}{4}(4H^3 + 6HX_0^2 + 6\sqrt{3}HX_0Y_0 + 3X_0Y_0^2 - X_0^3) \tag{5}$$

As an example the expressions derived, will be evaluated for the following static parameters:

- $T_{\mathrm{REV}} = 0.5\ \mu\mathrm{s}$
- Spill length = 0.5 s = $10^6\ T_{\mathrm{REV}}$
- Initial $\delta Q = 3\ 10^{-3}$
- $\varepsilon = 6\pi\delta Q = 5.65\ 10^{-2}$
- λ[†] $= 3\ 10^{-6}$
- $S = 36.7\ \mathrm{m}^{-1/2}$
- $H = (2/3)(\varepsilon/S) = 1.03\ 10^{-3}\ \mathrm{m}^{1/2}$

2.1 Extraction time when starting near stable point O´ (0 < Λ < 0.1)

In the vicinity of the fixed point O´, that is $|X|, |Y| << H$, the third order terms in X and Y can be neglected. Simplifying, the trajectory equation (5) results in,

$$Y = \frac{X_0^2 + \sqrt{3}X_0Y_0 - X^2}{\sqrt{3}X} \tag{6}$$

[†] The value of λ is chosen to correspond to the distance moved by the separatrix during 3 turns in the example shown in the next sections, when acceleration is taken into account. Its meaning will

where X_0 and Y_0 are the coordinates of the starting point. The substitution of this trajectory into the equation of motion for X, equation (4) (a), yields:

$$\frac{dX}{dt}=\frac{S}{4}\left(6\sqrt{3}HX+\frac{6}{\sqrt{3}}X_0^2+6X_0Y_0-\frac{6}{\sqrt{3}}X^2\right) \tag{7}$$

This equation can be integrated by variable separation. By definition, the trajectory equation (6) is valid only close to O´, but as the particle approaches asymptotically the separatrix $Y=-\frac{1}{\sqrt{3}}X$ the third order terms in the hamiltonian cancel out, so that they can also be neglected far from O´ along the outgoing separatrix and integration can be extended to the electrostatic septum. Using the standard integral formula:

$$\int\frac{1}{ax^2+bx+c}dx=\frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| \tag{8}$$

valid for $b^2>4ac$, the transit time T_c can be written as

$$T_c=\int_{t_0}^{t_0+T_c}dt=\frac{2}{\sqrt{3}S}\int_{X_0}^{X_{ES}}\frac{1}{-X^2+3HX+X_0^2+\sqrt{3}X_0Y_0}dX=$$

$$=\frac{2}{\sqrt{3}S}\frac{1}{\sqrt{9H^2+4\left(X_0^2+\sqrt{3}X_0Y_0\right)}}\left[\ln\left|\frac{-2X+3H-\sqrt{9H^2+4\left(X_0^2+\sqrt{3}X_0Y_0\right)}}{-2X+3H+\sqrt{9H^2+4\left(X_0^2+\sqrt{3}X_0Y_0\right)}}\right|\right]_{X_0}^{X_{ES}}$$

Since $|X_0|$, $|Y_0| << H$, we can expand the square roots to first order:

$$T_c\approx\frac{2}{3\sqrt{3}SH}\left[\ln\left|\frac{-2X+3H-3H\left(1+\frac{2}{9}\frac{X_0^2+\sqrt{3}X_0Y_0}{H^2}\right)}{-2X+3H+3H\left(1+\frac{2}{9}\frac{X_0^2+\sqrt{3}X_0Y_0}{H^2}\right)}\right|\right]_{X_0}^{X_{ES}}=$$

$$\approx\frac{2}{3\sqrt{3}SH}\left[\ln\left|\frac{-X-\frac{X_0^2+\sqrt{3}X_0Y_0}{3H}}{-X+3H+\frac{X_0^2+\sqrt{3}X_0Y_0}{3H}}\right|\right]_{X_0}^{X_{ES}}$$

It is useful to express all the distances in units of H, thus we define

$$X_{ES}=-nH,$$

$$X_0 = -\lambda H,$$
$$Y_0 = -2\sqrt{3}\Lambda H .$$

For λ = 0, Λ = 0 corresponds to the upper corner and Λ = 1 to the lower corner of the stable triangle. For the position of the ES, n = 7.79 has been chosen for the numerical example.

$$T_c \approx \frac{2}{3\sqrt{3}SH} \ln \left| \frac{nH - \frac{1}{3}\left(\lambda^2 + \sqrt{3}\lambda\Lambda\right)H}{nH + 3H + \frac{1}{3}\left(\lambda^2 + \sqrt{3}\lambda\Lambda\right)H} \frac{\lambda H + 3H + \frac{1}{3}\left(\lambda^2 + \sqrt{3}\lambda\Lambda\right)H}{\lambda H - \frac{1}{3}\left(\lambda^2 + \sqrt{3}\lambda\Lambda\right)H} \right| =$$

$$\approx \frac{2}{3\sqrt{3}SH} \ln \left| \frac{n}{n+3} \frac{3}{\lambda} \right|$$

where, inside the logarithm, the term $\frac{2}{3}\left(\lambda^2 + \sqrt{3}\lambda\Lambda\right)H$ has been neglected with respect to nH in the first fraction and with respect to λH in the denominator of the second fraction (since Λ is small near to O´). In the numerator of the second fraction only the term 3H has been kept.

Remembering that $H = \frac{2}{3}\frac{\varepsilon}{S}$, it results that

the extraction time from the vicinity of O´ in the static case is

$$T_c(\lambda) \approx \frac{1}{\sqrt{3}\varepsilon} \ln \left| \frac{n}{n+3} \frac{3}{\lambda} \right| \quad (9)$$

Note that, provided $\Lambda << 1$, the time needed for a particle to be extracted is independent of the initial Y coordinate. For the numerical data given,

$$T_c = 137.7$$

which means that the particle needs 413 revolutions to reach the ES. A numerical simulation of the extraction process, yields 444 revolutions for $\Lambda = 0.1$ and 438 revolutions for $\Lambda = 0$. The simulation just advances the particle, applying the displacement that it undergoes each third revolution, see Appendix A, until it reaches the ES. The number of iterations multiplied by 3, gives the number of revolutions.

2.2 Time spent along the side of the stable triangle

For a particle starting somewhere along the side of the stable triangle, far from the stable point O´, the extraction time is given by the sum of the time needed to travel to O´ and the time needed to go from O´ to the ES. The motion along the side of the triangle is determined by the second equation of motion (4) (*b*). The trajectory of a particle starting near the side of the stable triangle is not a straight line but it remains close to the side for most of its length. Until $|X| << |Y|$ it is possible to approximate equation (4) (*b*), by neglecting the terms in X and X^2, so that,

$$\frac{dY}{dt} \approx -\frac{S}{4}\left(6\sqrt{3}HY + 3Y^2\right) \tag{10}$$

which, similar to the previous case, can be integrated by variable separation, yielding:

$$T_{s1} \approx \frac{1}{\sqrt{3}\varepsilon}\ln\left|\frac{\Lambda}{1-\Lambda}\frac{2\sqrt{3}-A}{A}\right| \tag{11}$$

where the integration has been done between the starting point $Y_0 = -2\sqrt{3}\Lambda H$ and $Y_F = -AH$, a point near the corner where the curve will be joined to the trajectory calculated in the previous Section 2.1. As the junction point has to be near O´ to allow the use of (9), A has to be smaller or of the order of $0.1 \cdot 2\sqrt{3}$, which corresponds to $\Lambda = 0.1$.

2.3 Extraction time for particle starting far from O´ (0.1 < Λ < 1)

To correctly join the two trajectories, it is necessary to evaluate the position of the start of the second. To do this, we can evaluate the hamiltonian (3) at the starting point $\left(-\lambda H, -2\sqrt{3}\Lambda H\right)$ and work out the X_F coordinate at $Y_F = -AH$. From

$$\frac{S}{4}(4H^3 + 6HX_F^2 + 6\sqrt{3}HX_F(-AH) + 3X_F(-AH)^2 - X_F^3) =$$

$$= \frac{S}{4}(4H^3 + 6H(-\lambda H)^2 + 6\sqrt{3}H(-\lambda H)\left(-2\sqrt{3}\Lambda H\right) + 3(-\lambda H)\left(-2\sqrt{3}\Lambda H\right)^2 - (-\lambda H)^3)$$

neglecting the higher order terms in λ and X_F, X_F is obtained,

$$X_F = -\frac{12\Lambda(1-\Lambda)}{A\left(2\sqrt{3}-A\right)}\lambda H \approx -\frac{12\Lambda(1-\Lambda)}{2A\sqrt{3}}\lambda H = -\lambda_F H \tag{12}$$

Starting from this point it is possible to evaluate the total time needed to reach the electrostatic septum:

$$T_s = T_{s1} + T_c \approx \frac{1}{\sqrt{3}\varepsilon}\ln\left|\frac{\Lambda}{1-\Lambda}\frac{2\sqrt{3}-A}{A}\frac{n}{n+3}\frac{3}{\lambda_F}\right| \approx \frac{1}{\sqrt{3}\varepsilon}\ln\left|\frac{\Lambda}{1-\Lambda}\frac{2\sqrt{3}}{A}\frac{n}{n+3}\frac{3}{\lambda_F}\right|$$

And finally

the extraction time starting far from O´ in the static case is

$$T_s(\lambda, \Lambda) \approx \frac{1}{\sqrt{3}\varepsilon}\ln\left|\frac{1}{(1-\Lambda)^2}\frac{n}{n+3}\frac{3}{\lambda}\right|, \tag{13}$$

which does not depend on *A*, that is on the exact position of the junction point between the two trajectories. This expression is valid until (1–Λ) >> λ, such that the velocity in *Y* does not depend on *X*.

With the numerical example given:

Λ	Formula (13) (revolutions)	Numerical Simulation (revolutions)
0.1	420	444
0.5	456	480
0.9	554	585
0.999	836	894

In Appendix B, an alternative formula for the time needed from the lower corner is presented.

Thus, two expressions, (9) and (13), have been found which predict the time needed for a particle to be extracted under static conditions.

3. EXTRACTION TIME WITH VARYING PARAMETERS

Resonant slow extraction can be "activated" in many ways, but the most common is to vary the tune Q of the machine, by varying the focusing quadrupoles, to bring the beam into resonance. Alternatively, it is possible to vary the momentum of the particles, as $\delta Q = Q' \, \Delta p/p$. Varying δQ causes the triangle to shrink, in fact $\varepsilon = 6\pi\delta Q$ is directly proportional to the apothem H of the stable triangle. Reducing the size of the triangle causes some particles to pass from the stable region inside the triangle to the unstable region outside and thus to be extracted. In this Section a linear variation of the tune with time, that is $\dot{Q}$ = constant, will be considered.

The main approximations used are the following:

- the relative variation in the size of the triangle during the extraction time will be small, that is $\Delta H << H$;

- instead of considering the movement of the separatrices as the triangle shrinks, the relative movement of the separatrix to the particle, will be considered as an additional contribution to the particle velocity and the triangle will be considered fixed during the extraction time. This approximation is illustrated in Figure 3, where the situation is sketched at two different times t_1 and t_2, with $t_2 > t_1$.

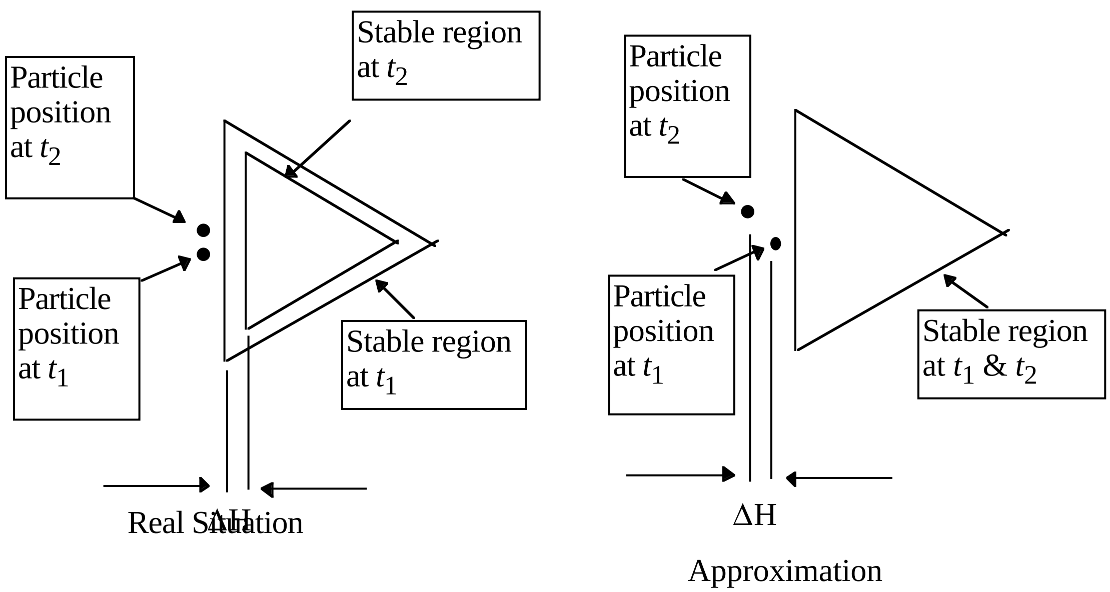

Figure 3. The relative motion of the particle and the stable triangle.

3.1 Extraction time for particle starting near point O´ ($0 < \Lambda < 0.1$)

It was shown in the static case, that in this region the extraction time does not depend on the initial value Λ (see (9)). Following this hint, Λ is set to zero and the particle is assumed to move on the outgoing separatrix $Y = -\frac{1}{\sqrt{3}}X$. With this assumption, equation (4) (a) becomes:

$$\frac{dX}{dt} = \frac{\sqrt{3}S}{2}\left(3HX - X^2\right).$$

When considering acceleration, it is necessary to add the velocity of the separatrix as it recedes from the particle. It is easy to evaluate it in the non translated frame and to note that the velocity has to be the same in both frames. In this frame, the separatrix equation is given by $X = -H$. Thus:

$$\frac{dH}{dt} = -\frac{4\pi}{S}\frac{dQ}{dt} \tag{14}$$

As the separatrix moves to the right, the particle moves to the left with respect to it. Thus, the relative velocity of the particle with respect to the separatrix, is given by:

$$\frac{dX}{dt} = \frac{\sqrt{3}S}{2}\left(3HX - X^2\right) + \frac{4\pi}{S}\dot{Q} \tag{15}$$

This equation can again be integrated by separation of variables and the extraction time results:

$$T_{c,a}\left(\lambda, \dot{Q}\right) = \frac{1}{\sqrt{3\varepsilon^2 + 8\pi\sqrt{3}\dot{Q}}}\left[\ln\left|\frac{-\sqrt{3}SX + \sqrt{3}\varepsilon - \sqrt{3\varepsilon^2 + 8\pi\sqrt{3}\dot{Q}}}{-\sqrt{3}SX + \sqrt{3}\varepsilon + \sqrt{3\varepsilon^2 + 8\pi\sqrt{3}\dot{Q}}}\right|\right]_{-\lambda H}^{-nH} =$$

$$\approx \frac{1}{\sqrt{3}\varepsilon}\left[\ln\left|\frac{-\sqrt{3}SX + \sqrt{3}\varepsilon - \sqrt{3}\varepsilon\left(1 + \frac{1}{2}\frac{8\pi\sqrt{3}\dot{Q}}{3\varepsilon^2}\right)}{-\sqrt{3}SX + \sqrt{3}\varepsilon + \sqrt{3}\varepsilon\left(1 + \frac{1}{2}\frac{8\pi\sqrt{3}\dot{Q}}{3\varepsilon^2}\right)}\right|\right]_{-\lambda H}^{-nH} =$$

$$= \frac{1}{\sqrt{3}\varepsilon}\ln\left|\frac{\left(\frac{2}{\sqrt{3}}n - 4\pi\frac{\dot{Q}}{\varepsilon^2}\right)}{\left(\frac{2}{\sqrt{3}}n + 2\sqrt{3} - 4\pi\frac{\dot{Q}}{\varepsilon^2}\right)}\frac{\left(\frac{2}{\sqrt{3}}\lambda + 2\sqrt{3} - 4\pi\frac{\dot{Q}}{\varepsilon^2}\right)}{\left(\frac{2}{\sqrt{3}}\lambda - 4\pi\frac{\dot{Q}}{\varepsilon^2}\right)}\right|$$

whence

the extraction time from the vicinity of O´ in the dynamic case is

$$T_{c,a}\left(\lambda,\dot{Q}\right) \approx \frac{1}{\sqrt{3}\varepsilon}\ln\left|\frac{n}{(n+3)}\frac{2\sqrt{3}}{\left(\frac{2}{\sqrt{3}}\lambda-\frac{4\pi\dot{Q}}{\varepsilon^2}\right)}\right| = \frac{1}{\sqrt{3}\varepsilon}\ln\left|\frac{n}{(n+3)}\frac{3}{\left(\lambda-\frac{1}{\sqrt{3}}\frac{\dot{\varepsilon}}{\varepsilon^2}\right)}\right| \qquad (16)$$

where the assumption $\Delta H << H$ which translates into:

$$\left|\frac{4\pi\dot{Q}}{S}\frac{1}{\sqrt{3}\varepsilon}\right| << \left|\frac{2}{3}\frac{\varepsilon}{S}\right| \Rightarrow \left|\dot{Q}\right| << \frac{1}{2\sqrt{3}\pi}\varepsilon^2$$

has been used and λ and $\frac{\dot{Q}}{\varepsilon^2}$ have been neglected with respect to unity. In principle, $\lambda = -\frac{\dot{\varepsilon}}{\varepsilon}$ is also negligible with respect to $\frac{\dot{\varepsilon}}{\varepsilon^2}$. The meaning is that particles start within one step of the separatrix and the separatrix moves many times during extraction. The term in λ has not been suppressed in (16) in order to recover (9) when $\dot{Q} = 0$.

If the numerical data proposed in Section 2.1 are used with the additional values:

- $\Delta\delta Q$ per turn = -3 $10^{-3}/10^{6}$ = -3 10^{-9}
- $\dot{Q} = \Delta\delta Q$ in 3 turns = -9 10^{-9}
- $\lambda = -\Delta H$ in 3 turns/H = $-\dot{Q}/(\delta Q)_0$ = 3 10^{-6}

then the transit time $T_{c,a}$ is 339 machine revolutions, compared to the numerical simulation result of 357 turns for Λ = 0, and 360 turns for Λ = 0.1.

3.2 Time spent along the side of the stable triangle

The next step is to evaluate the time needed to travel along the side of the stable triangle. The same approximations will be used as before, with the only difference that the velocity of the separatrix has now to be multiplied by $\frac{2}{\sqrt{3}}$ for geometrical reasons.

To be considered, is the extraction separatrix O´ to ES, which is moving downward in Figure 1. The side of the stable triangle is also moving, but the motion is quasi-independent of λ. The extraction separatrix moves with the same speed as before, but only the *Y* component of its motion is of interest. Equation (4)(*b*), provided that *X* remains negligible during the particle movement, becomes,

$$\frac{dY}{dt} = -\frac{S}{4}\left(6\sqrt{3}HY+3Y^2\right)-\frac{8\pi}{\sqrt{3}S}\dot{Q} \qquad (17)$$

which, integrated between $Y_0 = -2\sqrt{3}\Lambda H$ and $Y_{F,a} = -AH$, gives:

$$T_{s1,a}(\Lambda, A) \approx \frac{1}{\sqrt{3}\varepsilon} \ln \left| \frac{\Lambda}{1-\Lambda} \frac{2\sqrt{3}-A}{A} \right| \tag{18}$$

which is exactly the same as in the case of the static situation, equation (11). This is reasonable considering the approximation used, in which the velocity in Y does not depend on X and in which the variation in the length of the triangle side is negligible in the time considered.

3.3 Extraction time for particles starting far from O´ (0.1 < Λ < 1)

It only remains to evaluate the point where the second integral begins. The assumption is made that this point can be calculated by adding the position calculated under stationary conditions using (12), the first addendum in the following formula, and the movement of the separatrix, evaluated as the product of the time needed to move along the side of the triangle times the velocity at which the separatrix moves. The starting point for the second integral is then:

$$\begin{aligned} X_{F,a} &= -\frac{12\Lambda(1-\Lambda)}{A\left(2\sqrt{3}-A\right)}\lambda H + \frac{1}{\sqrt{3}\varepsilon}\ln\left|\frac{\Lambda}{1-\Lambda}\frac{2\sqrt{3}-A}{A}\right|\frac{4\pi}{S}\dot{Q} = -\lambda_{F,a} H \\ Y_{F,a} &= -AH \end{aligned} \tag{19}$$

The total time to reach the ES for a particle starting from $\left(-\lambda H, -2\sqrt{3}\Lambda H\right)$ with Λ greater or of the order of 0.1, taking into account the shrinking of the stable triangle, is given by the sum of the time to move along the side and the time to reach the ES from the junction point near O´, that is:

$$T_{s,a} = T_{s1,a} + T_{c,a} \approx \frac{1}{\sqrt{3}\varepsilon}\ln\left|\frac{\Lambda}{1-\Lambda}\frac{2\sqrt{3}-A}{A}\right| + \frac{1}{\sqrt{3}\varepsilon}\ln\left|\frac{n}{n+3}\frac{2\sqrt{3}}{\frac{2}{\sqrt{3}}\lambda_{F,a} - \frac{4\pi}{\varepsilon^2}\dot{Q}}\right|$$

And finally

the extraction time starting far from O´ in the dynamic case is

$$T_{s,a}\left(\lambda, \Lambda, \dot{Q}, A\right) \approx \frac{1}{\sqrt{3}\varepsilon} \ln \left| \frac{n}{n+3} \frac{2\sqrt{3}-A}{A} \frac{\Lambda}{1-\Lambda} \left(\frac{3}{\lambda_{F,a} - \frac{2\pi\sqrt{3}}{\varepsilon^2}\dot{Q}} \right) \right| = \tag{20}$$

$$= \frac{1}{\sqrt{3}\varepsilon} \ln \left| \frac{n}{n+3} \frac{2\sqrt{3}-A}{A} \frac{\Lambda}{1-\Lambda} \left(\frac{3}{\lambda_{F,a} - \frac{1}{\sqrt{3}} \frac{\dot{\varepsilon}}{\varepsilon^2}} \right) \right|$$

This time, A does not fortuitously disappear, as in the static case, thus a value must be given for it. As the two regions (near and far from O´) have been previously divided at $\Lambda = 0.1$, it seems reasonable to choose A such that the junction point corresponds to $\Lambda = 0.1$. This means $A = 0.1 \cdot 2\sqrt{3}$. Evaluating (19) and (20) with the numerical data used earlier, and comparing with the simulation, yields:

Λ	Formula (20) (revolutions)	Numerical Simulation (revolutions)
0.1	339	360
0.5	371	378
0.9	424	432
0.999	550	579

4. CONCLUSIONS

It has been possible to express the transit time in a third order resonance in an analytic form which agrees within a few percent with numerical simulations over the full range of practical parameters. The free parameter that determines the liaison between the trajectory following the side of the stable triangle and the trajectory following the extraction separatrix has been set to a single value $\frac{\sqrt{3}}{5}$. If equation (20) is used for all the cases, then the agreement is good to within 10%.

5. ACKNOWLEDGEMENTS

This work was performed at CERN, within the framework of the Proton-Ion Medical Machine Study. The author would like to thank the PS Division for its hospitality in hosting this study and to acknowledge all the friendly help and advice given by PS staff members. In particular, he would like to thank P. Bryant and J. Bosser for all the times he has bothered them with this study and S. Rossi for the many useful discussions.

APPENDIX A

The Kobayashi hamiltonian

The Kobayashi hamiltonian is derived by considering the displacement of a particle in normalized phase space after three revolutions of the machine (for a third integer resonance) and integrating the equations of motion which result. For a tune value Q close to an integer multiple of 1/3, the particle will turn about 120 (or 240) degrees in normalized phase space per revolution and after 3 revolutions it will return close to the starting position. Taking into account the kicks due to the sextupole to first order and considering the phase space at the sextupole, the motion equations are,

$$\frac{dX}{dt} = \Delta X_{3REV} = \varepsilon X'_0 + \frac{3}{2} S X_0 X'_0$$

$$\frac{dX'}{dt} = \Delta X'_{3REV} = -\varepsilon X_0 + \frac{3}{4} S\left(X_0^2 - X'^2_0\right)$$

where $\varepsilon = 6\pi\delta Q$, $S = -\beta^{3/2} \frac{1}{2} \frac{l_s}{|B\rho|} \left(\frac{d^2 B_z}{dx^2}\right)_0$. The normalized coordinates are given by

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

The normalized coordinates have the dimensions $m^{1/2}$ and the normalized sextupole strength $m^{-1/2}$. Note that since $\frac{dX}{dt} = \Delta X$, the quantity that is used for the time is dimensionless and equal to the number of times the particle completes 3 revolutions of the accelerator.

APPENDIX B

Time from the vicinity of point B, 2nd formula

On inspection of the numerical results, it appears that equation (13) is particularly accurate for $\Lambda = 0.5$. This suggests that it would be more accurate to evaluate the time needed to extract a particle from the region near the lower corner, T_{lc}, by:

$$T_{lc} = 2T(\lambda_{center}, \Lambda = 0.5) - T_c(\lambda) \tag{B1}$$

where λ_{center} is the λ value that the particle will have at the center of the side and evaluated as before using the fact that the hamiltonian has to be a constant of the motion.

$$\lambda_{center} = \frac{2\lambda^2 + 6\lambda(1-\Lambda)}{3}$$

The evaluation and simplification of (B1) yields:

$$T_{lc} = \frac{1}{\sqrt{3}\varepsilon}\ln\left(48\frac{\lambda}{\lambda_{center}^2}\frac{n}{n+3}\right) \tag{B2}$$

This expression is valid in the same range of Λ as equation (9) but for the lower corner, that is $(1-\Lambda) << 1$.

The fact that this expression works better than (13), is probably due to the fact that some of the approximations made cancel out in the subtraction.

References

1. M. Q. Barton, *Beam extraction from synchrotrons*, Proc. of the 8th Int. Conf. on High Energy Accelerators, CERN, Geneva (1971) p. 85.